UNIT- III

Electromagnetic Theory and Maxwell's Equations

* Ampere's Law and steady state current

Statement -

The line integral $\oint \bar{B}\cdot d\bar{l}$ for a closed curve is equal to μ_0 times the net current I passing through the area bounded by the curve. That is

$$\boxed{\oint \bar{B}\cdot d\bar{l} = \mu_0 I}$$

where μ_0 is the permeability constant.

Let $\bar{J}$ be the current density in an element $\bar{ds}$ of the surface bounded by the closed path. Then

$$\text{total current } I = \int_s \bar{J}\cdot\bar{ds}$$

$$\oint \bar{B}\cdot d\bar{l} = \mu_0 I = \mu_0 \int_s \bar{J}\cdot\bar{ds}$$

Using stokes theorem

$$\oint \bar{B}\cdot d\bar{l} = \int_s \text{curl}\,\bar{B}\cdot\bar{ds}$$

$$\therefore \int_s \text{curl}\,\bar{B}\cdot\bar{ds} = \mu_0 \int_s \bar{J}\cdot\bar{ds}$$

or $\quad \text{curl}\,\bar{B} = \mu_0 \bar{J}$

This is the differential form of Ampere's law.

* Generalization of Ampere's Law - Displacement current

The magnetic field due to a current distribution satisfied Amperes circuital law

$$\oint H\cdot dl = \int_s J\cdot n\, da \quad \text{---} (1)$$

This equation sometimes fails, we find generalization that is always valid

consider a circuit shown in below fig. (a).

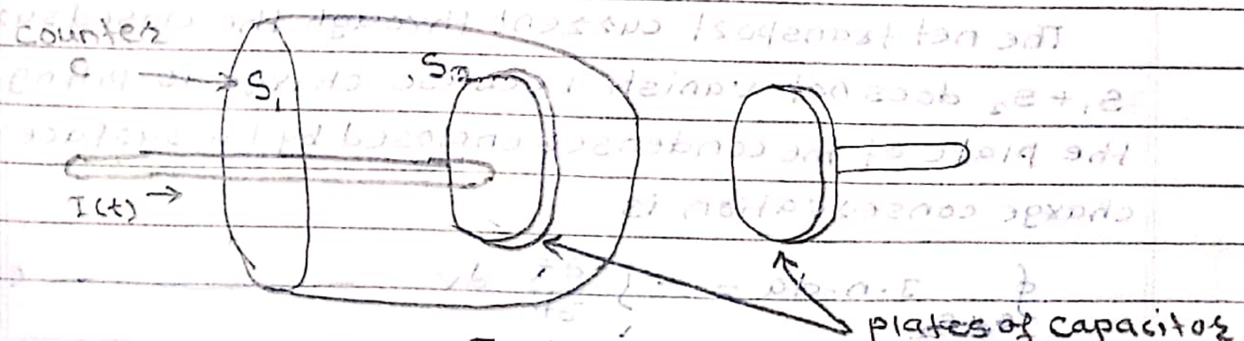

Fig (a)

The counter C and two surfaces S_1 & S_2 for testing Ampere's Law

The figure consist of a small parallel-plate capacitor being charged by a constant current I. If Ampere's law is applied to the counter c and surface s, we find

$$\oint H \cdot dl = \int_{S_1} J \cdot n \cdot da = I \quad \text{——— (2)}$$

If Ampere's law is applied to the counter c and surface S_2 then J is zero at all the points on S_2 and

$$\oint H \cdot dl = \int_{S_2} J \cdot n\, da = 0 \quad \text{——— (3)}$$

Equations (2) and (3) contradict each other and thus can not both be true or correct. If c is imagined to be a great distance from the capacitor, it is clear that the situation is not substantially different from the standard Ampere's law. so eqⁿ (3) require modification

$$\nabla \times H = J \quad \text{——— (4)}$$

The proper modification can be made by noting that eqⁿ (2) and (3) give different results because the integrals on the right-hand side are different.

$$\therefore \int_{S_2} J \cdot n_2\, da = \int_{S_1} J \cdot n_1\, da \neq 0 \quad \text{——— (5)}$$

S_1 and S_2 are together form a closed surface, n_2 is outward drawn and n_1 is inward drawn. If this fact is taken into account eqⁿ (5) may be written as

$$\oint_{S_1+S_2} J \cdot n \cdot da \neq 0 \quad \text{——— (6)}$$

The net transport current through the closed surface S_1+S_2 does not vanish because charge is piling upto the plate of the condenser enclosed by the surface charge conservation is

$$\oint_{S_1+S_2} J \cdot n \cdot da = -\int_V \frac{\partial \rho}{\partial t}\, dv \quad \text{——— (7)}$$

because inside the volume V enclosed by $S_1 + S_2$ the charge density ρ is changing with time on the condenser plate. In differential form eqn (7) is expressed by the equation of continuity

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0 \quad \text{---------- (8)}$$

Taking the divergence of eqn (4) $\nabla \cdot \nabla \times H = \nabla \cdot J = 0$ since the divergence of curl is zero. Thus the relation $\nabla \cdot J = 0$, as implied by eqn (4) is inconsistent with charge conservation in the present situation, and something must be added to the right side of eqn (4) that will be $\partial\rho/\partial t$ in eqn (8). We will see this from the relation of ρ to electric displacement

$$\nabla \cdot D = \rho \quad \text{---------- (9)}$$

Inserting ρ from eqn (9) into eqn (8) we get

$$\nabla \cdot J + \frac{\partial}{\partial t}(\nabla \cdot D) = \nabla \cdot \left(J + \frac{\partial D}{\partial t} \right) = 0$$

If $\partial D/\partial t$ is added to the equation (4), then its divergence will correctly give eqn (8). Inclusion of $\partial D/\partial t$ gives the generalised Ampere's law

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad \text{---------- (10)}$$

The introduction of the second term on the right which is known as the displacement current represents the Maxwell's major contributions to the electromagnetic theory.

* Maxwell's Equations

The set of Maxwell's equations are given below

$$\nabla \cdot \bar{B} = 0 \quad \text{——— (1)}$$

$$\nabla \cdot \bar{D} = \rho \quad \text{——— (2)}$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{——— (3)}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{——— (4)}$$

Each of these equations represents a generalization of certain experimental observations.

Equation (1) is usually said to represent the fact that single magnetic pole have never been observed.

Equation (2) represents Gauss law, which in turn derives from coulomb's law

Equation (3) represents the differential form of Faraday's law of Electromagnetic induction.

Equation (4) represents an extension of Amperes law.

* Derivation of Maxwell's Equations

(I) $\nabla \cdot \bar{B} = 0$

Since isolated magnetic poles have no physical significance. Magnetic lines of force in general are either closed curve or they goes to infinity. As a result number of magnetic lines of force entering any arbitrary closed surface is exactly the same leaving it. This means that total magnetic flux across any closed surface is zero.

Magnetic flux ϕ is given by

$$\phi = \int_s \bar{B} \cdot d\bar{s}$$

using Gauss divergence theorem

$$\int_s \bar{B} \cdot d\bar{s} = \int_v \nabla \cdot \bar{B} \, dv$$

where v is surface bounded volume

$$\therefore \boxed{\nabla \cdot \bar{B} = 0}$$

⟨II⟩ $\nabla \cdot \bar{D} = \rho$

Let us consider a surface s, bounding a small volume υ in a dielectric medium. In a dielectric medium total charge consists of free charge plus polarization charge.

Let ρ = free charge density

ρ_p = polarization charge density

Density at a point in a volume element dυ is given by Gauss law

$$\int_S \bar{E} \cdot d\bar{s} = \frac{1}{\varepsilon_0} \int_V (\rho + \rho_p)\, d\upsilon \quad \text{—— (1)}$$

But polarization charge density is

$$\rho_p = -\nabla \cdot P \quad \text{—— (2)}$$

where P is polarization.

Using Gauss divergence theorem we can change surface integral of eq^n.(1) into volume integral and we have

$$\int_S \bar{E} \cdot d\bar{s} = \int_V \nabla \cdot (\varepsilon_0 \bar{E} d s)\, d\upsilon = \int_V (\rho - \nabla \cdot P)\, d\upsilon$$

$$\therefore \int_V \nabla \cdot (\varepsilon_0 \bar{E} + P)\, d\upsilon = \int_V \rho\, d\upsilon \quad \text{—— (3)}$$

we define

$$\varepsilon_0 E + P = D = \text{electric displacement}$$

$$\int_V \nabla \cdot D\, d\upsilon = \int_\upsilon \rho\, d\upsilon$$

$$\int_V [\nabla \cdot D - \rho]\, d\upsilon = 0$$

But $d\upsilon \neq 0$

so $\nabla \cdot D - \rho = 0$

$$\therefore \nabla \cdot \bar{D} = \rho$$

$$\boxed{\nabla \cdot \bar{D} = \rho}$$

⟨III⟩ $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

According to the Faraday's law of electromagnetic Induction

$$\mathcal{E} = -\frac{d\phi}{dt} \qquad \text{—— (1)}$$

This result is found to be independent of the way in which the flux is changed.

since by the definition

$$\mathcal{E} = \oint \bar{E} \cdot d\bar{l} \qquad \text{—— (2)}$$

and

$$\phi = \int_s \bar{B} \cdot \bar{n}\, da \qquad \text{—— (3)}$$

put (2) and (3) in equation (1)

$$\oint \bar{E} \cdot d\bar{l} = -\frac{d}{dt}\int_s \bar{B} \cdot \bar{n}\, da \qquad \text{—— (4)}$$

If the circuit is rigid stationary circuit, the time derivative can be taken inside integral, where it becomes partial derivative.

$$\oint_c \bar{E} \cdot d\bar{l} = -\int_s \frac{\partial \bar{B}}{\partial t}\, \bar{n}\, da$$

Using Stoke's theorem, we get

$$\int_s \nabla \times \bar{E}\, \bar{n} \cdot da = -\int_s \frac{\partial \bar{B}}{\partial t}\, \bar{n}\, da \qquad \text{—— (5)}$$

This must be true for all the surfaces.

It follows that

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \text{—— (6)}$$

$$\text{curl}\, \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

hence electromagnetic induction expression.

(IV) $\nabla \times H = J + \frac{\partial D}{\partial t}$

The Ampere's law is

$$\nabla \times \bar{B} = \mu_0 J \quad \text{——— (1)}$$

this does not hold good for time-varying fields. To remove this difficulty, Maxwell modified the above equation by introducing the concept of displacement current.

But putting $\bar{B} = \mu_0 \bar{H}$ in above eq^n we get

$$\nabla \times \mu_0 \bar{H} = \mu_0 \bar{J}$$

$$\therefore \nabla \times \bar{H} = \bar{J} \quad \text{——— (2)}$$

Taking divergence of this equation $\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J}$

But the divergence of the curl of a vector is always zero.

$$\therefore \nabla \cdot \bar{J} = 0 \quad \text{——— (3)}$$

This means that the divergence of current density is zero. This is not true for time-varying fields.

From Gauss's law,

$$\nabla \cdot \bar{E} = \frac{\rho}{\varepsilon_0} \quad \text{——— (4)}$$

Differentiating

$$\nabla \cdot \frac{\partial \bar{E}}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \rho}{\partial t}$$

or

$$\varepsilon_0 \nabla \cdot \frac{\partial \bar{E}}{\partial t} = \frac{\partial \rho}{\partial t} \quad \text{——— (5)}$$

Adding $\nabla \cdot \bar{J}$ to both sides of above eq^n (5) we have

$$\nabla \cdot \bar{J} + \varepsilon_0 \nabla \cdot \frac{\partial \bar{E}}{\partial t} = \nabla \cdot \bar{J} + \frac{\partial \rho}{\partial t}$$

According to general equation of continuity,

$$\nabla \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\therefore \nabla \cdot \bar{J} + \varepsilon_0 \nabla \cdot \frac{\partial \bar{E}}{\partial t} = 0$$

$$\therefore \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \varepsilon_0 \bar{E}}{\partial t} = 0 \quad \text{——— (6)}$$

But $\bar{D} = \varepsilon_0 \bar{E}$

$$\therefore \nabla \cdot \left(J + \frac{\partial D}{\partial t} \right) = 0$$

$(\bar{J} + \partial D/\partial t)$ is the total current density. Maxwell pointed out that this should replace $\bar{J}$ in Ampere's law. Hence the modified form of Ampere's law is

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \quad \text{--- (7)}$$

The first term on the R.H.S. of above equation represents the conduction current density $\bar{J}$.

The term $\partial \bar{D}/\partial t$ is called displacement current. It is the time rate of change of the electric displacement. The second term is noted as J_d i.e displacement current density. So, equation (7) becomes

$$\nabla \times \bar{H} = (J + J_d) \quad \text{--- (8)}$$

* The Electromagnetic Energy and Poynting Vector

As em waves propogate through matter from their source to distant receiving point, there is a transfer of energy from source to receiver. The rate at which energy is transmitted through unit area ⊥r to the direction of propogation of energy is called poynting vector and is denoted by P.

Two of the Maxwell's equations are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{--- (1)}$$

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \quad \text{--- (2)}$$

Taking scalar product of eqⁿ (1) with $\bar{H}$

$$\bar{H} \cdot (\nabla \times \bar{E}) = \bar{H} \cdot \left(-\frac{\partial \bar{B}}{\partial t}\right)$$

But $\bar{B} = \mu \bar{H}$

$$\therefore \quad \bar{H}\cdot(\nabla\times\bar{E}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) \qquad (3)$$

Taking scalar product of eq^n (2) with $\bar{E}$

$$\bar{E}\cdot(\nabla\times\bar{H}) = \bar{E}\cdot\bar{J} + \bar{E}\cdot\frac{\partial\bar{D}}{\partial t}$$

But $\bar{D} = \varepsilon\bar{E}$ so,

$$\bar{E}\cdot(\nabla\times\bar{H}) = \bar{E}\cdot\bar{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) \qquad (4)$$

subtracting equation (3) from eq^n (4), we get

$$\bar{E}\cdot(\nabla\times\bar{H}) - \bar{H}\cdot(\nabla\times\bar{E}) = \bar{E}\cdot\bar{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\bar{E}\cdot(\nabla\times\bar{H}) - \bar{H}\cdot(\nabla\times\bar{E}) = \bar{E}\cdot\bar{J} + \frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] \qquad (5)$$

But $\nabla\cdot(\bar{E}\times\bar{H}) = \bar{H}\cdot(\nabla\times\bar{E}) - \bar{E}\cdot(\nabla\times\bar{H})$

$$\therefore \quad -\nabla\cdot(\bar{E}\times\bar{H}) = \bar{E}\cdot\bar{J} + \frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] \qquad (6)$$

Let us consider a volume V enclosed by a surface S. Integrating the above relation over the volume V, we have

$$\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)dv + \int_V(\bar{E}\cdot\bar{J})\,dv = -\int_V \nabla\cdot(\bar{E}\times\bar{H})\,dv$$

Using Gauss divergence theorem

$$\int_V \nabla\cdot(\bar{E}\times\bar{H})\,dv = \oint_S(\bar{E}\times\bar{H})\cdot ds$$

$$\therefore \quad -\frac{\partial}{\partial t}\int_V\left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right)dv - \int_V(\bar{E}\cdot\bar{J})\,du = \oint_S(\bar{E}\times\bar{H})\,ds \qquad (7)$$

The first term on L.H.S. of eq^n (7) represents the rate of decrease of energy stored in volume V due to electric and magnetic fields. The second term on L.H.S. represents the rate at which electromagnetic energy is lost through Joule heating. Hence the R.H.S. represents the rate of flow of energy over the surface S enclosing volume V.

Therefore, $\bar{E}\times\bar{H}$ gives the rate of flow of energy through unit area enclosing the volume V. This is denoted by P, called Poynting vector.

$$\therefore \quad \boxed{P = \bar{E}\times\bar{H}} \qquad (8)$$

The direction P is ⊥r to both $\bar{E}$ and $\bar{H}$.

* The wave Equation for E and B

Let us apply Maxwell's equations to develop wave equations for transverse electric and magnetic fields in free space. Let us consider a region where charge density ρ and current density J both are zero, i.e for free space,

$$\mu = \mu_0, \quad \varepsilon = \varepsilon_0, \quad \rho = 0 \text{ and } \bar{J} = 0$$

Maxwell's equations then reduces to

$$\nabla\cdot\bar{E} = 0 \qquad (1)$$

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \qquad (2)$$

$$\nabla\cdot\bar{B} = 0 \qquad (3)$$

$$\nabla\times\bar{B} = \varepsilon_0\mu_0\frac{\partial\bar{E}}{\partial t} \qquad (4)$$

Applying curl to eq^n (2)

$$\nabla\times(\nabla\times\bar{E}) = \nabla\times\left(-\frac{\partial\bar{B}}{\partial t}\right)$$

$$\therefore \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}(\nabla\times\bar{B})$$

$$\therefore \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t}\right)$$

$$\therefore \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\mu_0\varepsilon_0\frac{\partial^2\bar{E}}{\partial t^2}$$

~~when~~

$$\nabla(\nabla\cdot\bar{E}) = \nabla^2\bar{E} - \mu_0\varepsilon_0\frac{\partial^2\bar{E}}{\partial t^2} \qquad (5)$$

when $\nabla\cdot\bar{E} = 0$ then

$$\boxed{\nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2}} \quad \text{------ (6)}$$

Applying curl to eqⁿ (4)

$$\nabla \times (\nabla \times \bar{B}) = \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}\right)$$

$$\therefore \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \bar{E})$$

$$\therefore \nabla(\nabla \cdot \bar{B}) - \nabla^2 \bar{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}$$

$$\therefore \nabla(\nabla \cdot \bar{B}) = \nabla^2 \bar{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2} \quad \text{------ (7)}$$

when $\nabla \cdot \bar{B} = 0$ then,

$$\boxed{\nabla^2 \bar{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{B}}{\partial t^2}} \quad \text{------ (8)}$$

Equations (6) and (8) represents wave equations governing electromagnetic fields $\bar{E}$ and $\bar{B}$ in free space.

Equations (6) and (8) both have the form of the general wave equation, with a speed

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Now $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

$$\therefore c = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}}$$

$$= 2.998 \times 10^{8} \text{ m/s}$$

which is the speed of light in free space. It thus follows that the field vectors $\bar{E}$ and $\bar{B}$ can be propogated as waves in free space and they travel with the speed of light.

Thus the velocity of em waves in vacuum is the same as that of light in vacuum. This fact suggests that light is form of electromagnetic wave